

**AD-A271 435**



2

**PL-TR-92-2325**

**CONVERGENCE OF THE HERMITE  
WAVELET EXPANSION**

**G. v. H. Sandri**



**Boston University  
College of Engineering and  
Center for Space Physics  
Boston, MA 02215**

**November 1992**

**Final Report  
1 November 1988 - 31 October 1992**

**Approved for public release; distribution unlimited**



**PHILLIPS LABORATORY  
Directorate of Geophysics  
AIR FORCE SYSTEMS COMMAND  
HANSCOM AIR FORCE BASE, MA 01731-5000**

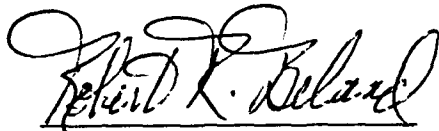
**93-25482**



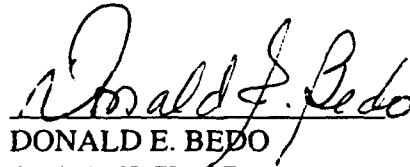
**93 10 21 079**

38145

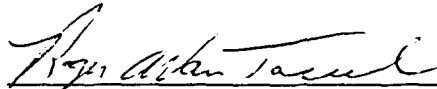
"This technical report has been reviewed and is approved for publication"



ROBERT R. BELAND  
CONTRACT MANAGER



DONALD E. BEDO  
BRANCH CHIEF



ROGER VAN TASSEL  
DIVISION DIRECTOR

This report has been reviewed by the ESD Public Affairs Office (PA) and is releasable to the National Technical Information Service (NTIS).

Qualified requestors may obtain additional copies from the Defense Technical Information Center. All others should apply to the National Technical Information Service.

If your address has changed, or if you wish to be removed from the mailing list, or if the addressee is no longer employed by your organization, please notify PL/TSI, Hanscom AFB, MA 01731-5000. This will assist us in maintaining a current mailing list.

Do not return copies of this report unless contractual obligations or notices on a specific document requires that it be returned.

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
<small>Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503</small>				
1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE November 1992	3. REPORT TYPE AND DATES COVERED Final. Nov. 1, 1988 to Oct. 31, 1992		
4. TITLE AND SUBTITLE  Convergence of the Hermite Wavelet Expansion		5. FUNDING NUMBERS  PE 61101F PR 7670 TA 15 WU AR Contract F19628-88-K-0017		
6. AUTHOR(S)  G. v. H. Sandri				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)  Boston University College of Engineering and Center for Space Physics Boston, MA 02215		8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)  Phillips Laboratory Hanscom AFB, MA 01731-5000  Contract Manager: Robert Beland/GPOA		10. SPONSORING/MONITORING AGENCY REPORT NUMBER  PL-TR-92-2325		
11. SUPPLEMENTARY NOTES				
12a. DISTRIBUTION/AVAILABILITY STATEMENT  Approved for public release; distribution unlimited.			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words)  <p>In this report we summarize the research carried out under this contract on the chaos dynamics analysis of the free sheared atmosphere. Our approach is to expand the fluid equations into finite energy modes rather than in the conventional Fourier analysis. We prove rigorously that our expansion method has the required convergence properties to ensure a satisfactory physical interpretation of the results. For the Taylor-Dyson atmosphere, our analysis, like the fourier analysis yields no unstable modes.</p>				
14. SUBJECT TERMS  Taylor-Dyson atmosphere, Hermite wavelet expansion.			15. NUMBER OF PAGES 38	
			16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT Unclassified	18. SECURITY CLASSIFICATION OF THIS PAGE Unclassified	19. SECURITY CLASSIFICATION OF ABSTRACT Unclassified	20. LIMITATION OF ABSTRACT SAR	

## CONTENTS

1.	INTRODUCTION	1
2.	MATHEMATICAL FRAMEWORK	2
3.	DEFINITIONS AND NOTATION	7
4.	CHARACTERIZATION OF HYPERDISTRIBUTIONS	11
5.	CLASSES $D(\{M_k\})$ AND THE DIFFUSION GROUP	21
6.	SUMMARY AND CONCLUSIONS	31
References		33

Accession For	
NTIS CRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution /	
Availability Codes	
Dist	Avail and/or Special
A-1	

# 1 Introduction

In this contract we have first investigated a phenomenological approach to a chaos dynamical analysis at the free sheered atmosphere. We have then turned to a theoretical phase in which we have developed a novel expansion of the equations for the free sheered atmosphere which hinge on properties of hermite polynomials and the gaussian function. In this final scientific report we give the conditions for the basic expansions used during our analysis to converge. More specifically, the necessary and sufficient conditions are given for the quasianalytic function classes  $D(\{M_k\})$  and the corresponding classes of distributions  $D'(\{M_k\})$  to be invariant with respect to the complex-time diffusion group

$$U_z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\partial^{2k}}{\partial x^{2k}}, \quad x \in R^1, z \in C.$$

In addition, the properties of hyperdistributions

$$\Gamma = \sum_{k=0}^{\infty} a_k \delta_0^{(k)}$$

are studied. It is shown that hyperdistributions are characterized by the behavior of their moments

$$\mu_k(\Gamma) = \Gamma(x^k).$$

We conclude that expansions based on hermite polynomials multiplied by a gaussian have a proper limiting behaviour. The objective to give complete mathematical foundations for our analysis has been accomplished.

## 2 Mathematical Framework

Many physical processes have diffusive character: the spreading of smoke in air, the behavior of the temperature in a material body, and the vorticity in a fluid flow are examples illustrating this feature. The engineer's notion of blurring and filtering have similar nature.

The one-dimensional diffusion processes are governed by the heat equation

$$\frac{\partial \Psi(x, t)}{\partial t} = \frac{\partial^2 \Psi(x, t)}{\partial x^2}, \quad x \in R^1, t > 0. \quad (1)$$

In order to determine the behavior of the physical quantity  $\Psi$  under consideration, provided its initial value

$$\Psi(x, 0) = \varphi(x), \quad x \in R^1, \quad (2)$$

is given, we have to solve the Cauchy problem (1) – (2) for the heat equation.

It is well-known that the solution of the Cauchy problem (1) – (2) with appropriately chosen initial condition  $\varphi$  is given by Poisson's formula

$$\Psi(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{R^1} \varphi(y) e^{-\frac{(x-y)^2}{4t}} dy, \quad x \in R^1, t > 0. \quad (3)$$

The following interpretation of formula (3) is possible. We get the solution  $\Psi(x, t)$  as a result of filtering the given data  $\varphi$  through a Gaussian filter of width  $\sqrt{t}$ .

Suppose we would like to reverse the process of filtering. This is important, for example, when we are concerned with the problem of reconstructing sharp images from degraded pictures. It is clear that the inverse filtering is described by the inverse heat equation

$$\frac{\partial \Psi(x, t)}{\partial t} = -\frac{\partial^2 \Psi(x, t)}{\partial x^2}, \quad x \in R^1, t > 0. \quad (4)$$

It is possible to incorporate the heat equation (1), the inverse heat equation (4), and the Schrödinger equation

$$\frac{\partial \Psi(x, t)}{\partial t} = i \frac{\partial^2 \Psi(x, t)}{\partial x^2}, \quad x \in R^1, t \in R^1.$$

which describes the one-dimensional motion of a free particle in quantum mechanics, into the so-called complex-time diffusion equation

$$\frac{\partial \Psi(x, z)}{\partial z} = \frac{\partial^2 \Psi(x, z)}{\partial x^2}, \quad x \in R^1, z \in C. \quad (5)$$

Consider now the following formal semi-group of operators

$$U_t = e^{t \frac{\partial^2}{\partial x^2}}, \quad t > 0 \quad (6)$$

(the diffusion semi-group). It can be easily seen [6] that the formula

$$\Psi(x, t) = U_t \varphi(x), \quad x \in R^1, t > 0$$

defines the formal solution of the Cauchy problem (1) – (2).

Using the Taylor expansion of the function  $e^{tu}$ ,  $u \in R^1$  in (6), we can represent  $\{U_t\}$  as a semi-group of differential operators of infinite order

$$U_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial^{2k}}{\partial x^{2k}}, \quad t > 0. \quad (7)$$

If we put a complex number  $z \in C$  instead of  $t$  in (7), we obtain the complex-time diffusion group of operators

$$U_z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\partial^{2k}}{\partial x^{2k}}, \quad z \in C. \quad (8)$$

The group  $\{U_z\}$  provides a formal solution of the complex-time diffusion equation (5). Moreover, it is a formal analytic continuation of the semi-group (7). We use the word “formal” in

considerations above, because we have not yet defined the domain of the operators  $U_t$  given by (8).

Suppose  $\{M_k\}$  is a sequence of positive numbers. Define a function class  $D(\{M_k\})$  on  $R^1$  by the following formula

$$D(\{M_k\}) = \{\varphi \in C^\infty(R^1) : |\varphi^{(k)}(x)| \leq Ah^k M_k, k \geq 0, x \in R^1\},$$

where positive constants  $A$  and  $h$  depend on  $\varphi$ . These classes are important in making the formal considerations above precise.

Perhaps, Hadamard [4], [5] was one of the first who understood the importance of classes, defined by given upper bounds for the successive derivatives of functions, in dealing with the Cauchy problem for the heat equation.

Hadamard posed in [4] the problem of characterizing those classes  $D(\{M_k\})$ , for which every function  $\varphi \in D(\{M_k\})$  can be uniquely determined by the sequence  $\varphi^{(k)}(x_0)$ ,  $k \geq 0$  for any given  $x \in R^1$ . Such classes  $D(\{M_k\})$  are called the quasianalytic classes. Hadamard's problem has been solved by Denjoy and Carleman (see Section 2 below, where we formulate the Denjoy-Carleman Theorem).

The classes  $D(\{M_k\})$  have become useful tools in complex analysis [8], [10] – [11], in the theory of distributions [2], [13], and in the theory of differential operators of infinite order [1]. The simple example of their usefulness in the Cauchy problem for the heat equation is given by the following. For an appropriately chosen class  $D(\{M_k\})$ , the functions

$$U_t \varphi(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial^{2k} \varphi(x)}{\partial x^{2k}}, \quad t > 0$$

are defined for every function  $\varphi \in D(\{M_k\})$  and

$$U_t \varphi(x) = \Psi(x, t), \quad x \in R^1, t > 0,$$



where  $\Psi$  is given by (3).

An important contribution to the application of the quasianalytic classes in the theory of partial differential equations is due to Gelfand and Shilov [2], [3]. They contributed substantially to the theory of distributions over quasianalytic classes and to the uniqueness and well-posedness problems for the heat equation with complex diffusion coefficient  $a$ , namely

$$\frac{\partial \Psi(x, t)}{\partial t} = a \frac{\partial^2 \Psi(x, t)}{\partial x^2}, \quad x \in R^1, a \in C.$$

Gelfand and Shilov found in [2] the classes of generalized distributions which provide solutions to these problems.

They wrote in [2]: "Applications of these spaces to the Cauchy problem in Vol. 3 will illustrate the well-known statement of Hadamard's on the relation between uniqueness theorems in the Cauchy problem on the one hand, and the theory of quasianalytic functions and the general theory of functions of a complex variable, on the other."

One of the problems we consider in this work is to characterize those classes  $D(\{M_k\})$ , which are invariant with respect to the complex-time diffusion group (8), which means that

$$U_z(D(\{M_k\})) \subset D(\{M_k\}), \quad z \in C.$$

We answer this question in Section 5. The  $U_z$ -invariance of  $D(\{M_k\})$  implies that we can diffuse, anti-diffuse, disperse, and anti-disperse<sup>1</sup>, staying in the same class.

It is easy to see that the operator  $U_z$  in (8) is a convolution operator, defined by the formula

$$U_z \varphi = \varphi * \Gamma_z,$$

---

<sup>1</sup>Dispersion and anti-dispersion correspond to the Schrödinger equation for the free particle.

where  $\Gamma_z$  is the Green's function for equation (5), namely

$$\Gamma_z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \delta_0^{(2k)} \quad (9)$$

The symbol  $\delta_0$  in (9) denotes the Dirac's delta function at 0.

The formal series, involving all the derivatives of the delta-function, namely

$$\Gamma = \sum_{k=0}^{\infty} a_k \delta_0^{(k)}, \quad (10)$$

are called hyperdistributions. They are highly singular objects and Schwartz's theory of distributions (see [9], [14], [2]) does not include them. The most appropriate theory, which involves hyperdistributions, is that of the distributions over the classes  $D(\{M_k\})$  (see [2], [3], [13]). In this theory the functions  $\varphi \in D(\{M_k\})$  are considered as test functions and the distributions are defined as bounded linear functionals on the class  $D(\{M_k\})$ , equipped with appropriate topology. In [13] the formal series (10) were considered as distributions over non-quasianalytic classes  $D(\{M_k\})$  (see [13], p. 51).

Hyperdistributions are of much use in image processing (see [6], [7]). They have been used for deblurring and compressing of images. This is easy to understand if we recall that a hyperdistribution

$$\Gamma_t = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \delta_0^{(2k)}, \quad t > 0 \quad (11)$$

is the Green's function for the inverse heat equation and thus we may reconstruct sharp images from damaged ones by convolving them with hyperdistributions (11).

This paper is organized as follows: In Section 3 the necessary definitions and known results are gathered. Section 4 is concerned with the structure of hyperdistributions. We prove (see Theorems 3 and 4) that, roughly speaking, the hyperdistributions are distributions over classes  $D(\{M_k\})$ , which have moments of all orders, and their moments should satisfy special

conditions. In Section 5 we give a characterization of the  $U_x$ -invariant classes  $D(\{M_k\})$  and the  $U_x$  invariant classes of distributions  $D'(\{M_k\})$  (see Theorems 5 and 6). As the corollary of Theorem 5 we get the following result. The quasianalytic classes  $D(\{k^{\frac{1}{2}\gamma}\})$ ,  $0 \leq \gamma < 1$ , are  $U_x$ -invariant, while the class  $D(\{k^{k/2}\})$  is not (see Corollaries 2 and 3).

### 3 Definitions and Notation

**Definition 1** (see [8],[11],[12]). Suppose  $\{M_k\}$  is a sequence of positive numbers. We say that an infinitely differentiable function  $\phi$  on the real line belongs to the class  $D(\{M_k\})$  if there exist positive constants  $A_\phi$  and  $h_\phi$ , depending on  $\phi$ , such that the following estimates hold for the successive derivatives  $\phi^{(k)}$  of the function  $\phi$ ,

$$|\phi^{(k)}(x)| \leq A_\phi h_\phi^k M_k, \quad x \in R^1.$$

**Definition 2** (see [8],[11],[12]). The class  $D(\{M_k\})$  is called quasianalytic if

$$\phi \in D(\{M_k\}), \quad \phi^{(k)}(x_0) = 0 \text{ for some } x_0 \in R^1 \text{ and all } k \geq 0 \implies \phi \equiv 0, \quad (12)$$

while all the classes  $D(\{M_k\})$ , which does not satisfy the quasianalyticity condition (12) are called non-quasianalytic.

It is easy to see that classes  $D(\{M_k\})$  are linear and dilation-invariant. The quasianalytic classes  $D(\{M_k\})$  cannot contain functions with compact support.

**Definition 3** (see [19]). In this definition we equip the class  $D(\{M_k\})$  with the locally convex topology. First we consider linear subclasses,

$$D_m(\{M_k\}) = \{\phi \in D(\{M_k\}) : |\phi^{(k)}(x)| \leq A_\phi m^k M_k, \quad k \geq 0, m \geq 1\}$$

of  $D(\{M_k\})$ . Each of the classes  $D(\{M_k\})$  is a Banach space with the norm defined by

$$\rho_m(\phi) = \sup_{x \in R^1} \sup_{k \geq 0} \frac{|\phi^{(k)}(x)|}{M_k m^k}.$$

It is clear that

$$D(\{M_k\}) = \bigcup_m D_m(\{M_k\}).$$

We equip  $D(\{M_k\})$  with the inductive limit topology with respect to the family of its subspaces  $D_m(\{M_k\})$ ,  $m \geq 1$  (see [12] for the definition of the inductive limit).

**Definition 4** (see [13]). The space  $D'(\{M_k\})$  of all bounded linear functionals on the locally compact space  $D(\{M_k\})$ , equipped with the strong topology (see [12]), will be called the space of distributions over the class  $D(\{M_k\})$ .

It is not difficult to prove that every band-limited function  $\phi \in L^2(R^1)$  belongs to the class  $D(\{M_k\})$  with  $M_k = 1$ ,  $k \geq 0$ . The band-limitedness means that the support of the Fourier transform of  $\phi$  is bounded. One more example is given by the Gaussian  $\phi(x) = e^{-x^2}$  which belongs to the class  $D(\{k^{k/2}\})$ . In the book by Mandelbrojt [11] (see p. 89) there are examples of functions  $\phi \in D(\{M_k\})$ , which do not belong to any proper subclass  $D(\{M'_k\})$  of  $D(\{M_k\})$ .

Our next goal is to introduce new classes  $\tilde{D}(\{M_k\})$ , which contain  $D(\{M_k\})$  and all the polynomials.

**Definition 5** For given sequence  $\{M_k\}$  consider a class  $\tilde{D}(\{M_k\})$  of infinitely differentiable functions  $\phi$  on the real line such that there exists a constant  $h_\phi$ , depending only on  $\phi$ , and for every finite interval  $I \subset R^1$  there exists a constant  $A_{I,\phi}$ , depending on  $I$  and  $\phi$ , for which

$$|\phi^{(k)}(x)| \leq A_{I,\phi} h_\phi^k M_k, \quad x \in I, \quad k \geq 0.$$

**Definition 6** We introduce the locally convex topology of the class  $\tilde{D}(\{M_k\})$  in the following way. Consider linear subspaces

$$\tilde{D}_m(\{M_k\}) = \{\phi \in \tilde{D}(\{M_k\}) : \rho_{I,m}(\phi) = \sup_{x \in I} \sup_{k \geq 1} \frac{|\phi^{(k)}(x)|}{m^k M_k} < \infty, \quad I \subset \mathbb{R}^1\}. \quad (13)$$

The family of semi-norms  $\rho_{I,m}$  generates the Frechet space topology on  $\tilde{D}(\{M_k\})$ . It is clear that

$$\tilde{D}(\{M_k\}) = \bigcup_m \tilde{D}_m(\{M_k\}).$$

We equip  $\tilde{D}(\{M_k\})$  with the inductive limit topology with respect to the family of its linear subspaces  $\{\tilde{D}_m(\{M_k\})\}$ ,  $m \geq 1$  (see [12] for the definition of the inductive limit).

Suppose  $\Gamma$  is a bounded linear functional on the space  $\tilde{D}(\{M_k\})$ . The next definition introduces the moments of such functionals.

**Definition 7** For the functional  $\Gamma$  as above, the numbers

$$\mu_k(\Gamma) = \Gamma(x^k), \quad k \geq 0 \quad (14)$$

are called the moments of  $\Gamma$ .

It is clear that  $D(\{M_k\}) \subset \tilde{D}(\{M_k\})$ . Thus, every bounded linear functional  $\Gamma$  on  $\tilde{D}(\{M_k\})$  belongs to the space  $D'(\{M_k\})$ .

**Definition 8** The space  $\tilde{D}'(\{M_k\})$  of all bounded linear functionals on the locally convex space  $D(\{M_k\})$ , equipped with the strong topology (see [12]), will be called the space of distributions over the class  $D(\{M_k\})$ , which have moments.

The quasianalyticity property of the class  $D(\{M_k\})$  depends on the behavior of the Ostrovski function

$$T(r) = \sup_{k \geq 0} \frac{r^k}{M_k}, \quad r > 0$$

(see [8]). For every sequence  $\{M_k\}$ , the new sequence  $\{\underline{M}_k\}$ , defined by

$$\ln \underline{M}_k = \sup_{r > 0} (k \ln r - \ln T(r)), \quad (15)$$

is called the convex logarithmic regularization of the sequence  $\{M_k\}$ . The sequence  $\{\ln \underline{M}_k\}$  is the largest convex sequence, minorizing the sequence  $\{\ln M_k\}$ . If the initial sequence  $\{M_k\}$  is logarithmically convex, namely if

$$M_k^2 \leq M_{k-1} M_{k+1}, \quad k \geq 1,$$

then  $\underline{M}_k = M_k$ ,  $K \geq 0$ .

The main result in the theory of quasianalytic classes is called the Denjoy-Carleman theorem (see [8]).

**Theorem 1 (Denjoy-Carleman)** *The following conditions are equivalent:*

1. *The class  $D(\{M_k\})$  is quasianalytic.*

2.

$$\int_0^\infty \frac{\ln T(r)}{1+r^2} dr = \infty.$$

3.

$$\sum_{k=0}^\infty \frac{\underline{M}_k}{\underline{M}_{k+1}} = \infty.$$

4.

$$\sum_{k=0}^\infty (\underline{M}_k)^{-1/k} = \infty.$$

The following theorem reduces the case of the general classes  $D(\{M_k\})$  to the case of classes with logarithmically convex defining sequences.

**Theorem 2 (Cartan–Gorny)** (*see [8]*) *For every positive sequence  $\{M_k\}$  we have*

$$D(\{M_k\}) = D(\{\underline{M}_k\}).$$

As we have already mentioned in the introduction, the formal infinite series,

$$\Gamma = \sum_{k=0}^{\infty} a_k \delta_0^{(k)}$$

will be called hyperdistributions. We have (formally) the following formula for the moments (14) of  $\Gamma$ :

$$\mu_k(\Gamma) = (-1)^k k! a_k, \quad k \geq 0 \quad (16)$$

and the following moment representation for  $\Gamma$ :

$$\Gamma = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k(r) \delta_0^{(k)} \quad (17)$$

In section 4 formulas (16) and (17) will be given exact meaning.

## 4 Characterization of hyperdistributions

The first result in this section provides conditions for a hyperdistribution

$$\Gamma = \sum_{k=0}^{\infty} a_k \delta_0^{(k)} \quad (18)$$

to be a distribution, belonging to the class  $\tilde{D}'(\{M_k\})$ .

**Theorem 3** Suppose a hyperdistribution (18) and a sequence  $\{M_k\}$  are given. If

$$\sum_{k=0}^{\infty} |a_k| M_k h^k < \infty \quad (19)$$

for every  $h > 0$ , then  $\Gamma \in \tilde{D}'(\{M_k\})$  and

$$a_k = (-1)^k \mu_k(\Gamma) (k!)^{-1}, \quad k \geq 0. \quad (20)$$

**Remark 1** Theorem 1 shows that for a given sequence  $\{M_k\}$  all hyperdistributions (18), satisfying condition (19), have moments  $\mu_k(\Gamma)$  of all orders. Moreover,

$$\sum_{k=0}^{\infty} \frac{|\mu_k(\Gamma)|}{k!} M_k h^k < \infty \quad (21)$$

for every  $h > 0$  and the moment representation formula

$$\Gamma = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \mu_k(\Gamma) \delta_0^{(k)} \quad (22)$$

is true for  $\Gamma$ .

We now formulate the main result of this section. It will be shown that for some sequences  $\{M_k\}$  the inverse to Theorem 3 holds.

The restriction for sequences  $\{M_k\}$  will be as follows: there exists a positive function  $\rho(h)$ ,  $h \geq 0$  and a positive sequence  $\{\tau_n\}$  such that

1.

$$\sum_{n=0}^{\infty} \mu_n \tau_n < \infty$$

2.

$$M_{n+m} h^n \leq \rho(h)^{m+1} (n!) \tau_n M_m, \quad m \geq 0, n \geq 0, h > h_0.$$

Then the following result holds.



**Theorem 4** Suppose a sequence  $\{M_k\}$  satisfies the conditions above. Then every distribution  $\Gamma \in \tilde{D}'(\{M_k\})$  is a hyperdistribution (18), for which (20), (21), and (22) hold.

**Remark 2** The conditions 1)-2) above and the similar conditions in Section 4 are useful in problems, which we consider in this paper. The conditions 1)-2) imply, on the one hand, the differentiability condition for the classes  $D(\{M_k\})$ , namely

$$M_{k+1} \leq c^k M_k$$

with some  $c > 0$ . This condition is necessary and sufficient for the differentiation invariance of the class  $D(\{M_k\})$  (see [19], p. 57). On the other hand, conditions 1)-2) above imply

$$h^k M_k (k!)^{-1} \rightarrow 0 \quad (23)$$

as  $k \rightarrow \infty$  for each  $h > 0$ . Condition (23) guarantees the convergence of Taylor series

$$\phi(x) = \sum_{j=0}^{\infty} \frac{\phi^{(j)}(0)}{j!} x^j, \quad \phi \in \tilde{D}(\{M_k\}) \quad (24)$$

everywhere on the real line. This can be shown by estimating the Lagrange form of the remainder of the series in (24). Similarly, all the Taylor series

$$\phi^{(k)}(x) = \sum_{j=0}^{\infty} \frac{\phi^{(j+k)}(0)}{j!} x^j, \quad k \geq 0, \phi \in \tilde{D}(\{M_k\}) \quad (25)$$

converge uniformly on all subintervals of  $R^1$ .

**Proof of Theorem 3.** If a hyperdistribution (18) is given and if a function  $\phi$  belongs to the class  $\tilde{D}(\{M_k\})$ , then (19) implies the absolute convergence of the series

$$\Gamma(\phi) = \sum_{k=0}^{\infty} (-1)^k a_k \phi^{(k)}(0).$$

Moreover, if  $I$  is any interval, for which  $0 \in I$ , we have

$$|\Gamma(\phi)| \leq \rho_{I,m}(\phi) \sum_{k=0}^{\infty} |a_k| m^k M_k, \quad m \geq 1, \quad (26)$$

where the semi-norms  $\rho_{I,m}$  are defined in (13).

It follows from (26) that the functional  $\Gamma$  is bounded on the space  $\tilde{D}_m(\{M_k\})$ ,  $m \geq 1$ . Hence, it is bounded on the inductive limit  $\tilde{D}(\{M_k\})$  of the spaces  $\tilde{D}_m(\{M_k\})$ ,  $m \geq 1$  (see [12] for the properties of the inductive limits).

Formula (20) follows easily from the definition of the moments.

Theorem 3 is proved.

Proof of Theorem 4. We will need the following lemma.

**Lemma 1** Suppose conditions 1)-2) hold for a sequence  $\{M_k\}$ . Then the Taylor series (24) of a function  $\phi \in \tilde{D}(\{M_k\})$  converges to  $\phi$  in the topology of the space  $\tilde{D}(\{M_k\})$ .

Proof. Consider a sequence of remainders of the Taylor series of  $\phi$ , namely

$$\phi_j(x) = \sum_{m=j}^{\infty} \frac{\phi^{(m)}(0)}{m!} x^m, \quad j \geq 0.$$

By Remark 2, the sequence  $\phi_j$  tends to 0 as  $j \rightarrow \infty$  uniformly on every interval. Moreover, we may differentiate  $k$ -times under the summation sign (see (25) in Remark 2).

Differentiating  $k$  times, we get

$$\phi_j^{(k)}(x) = \begin{cases} \sum_{m=0}^{\infty} \frac{\phi^{(m+k)}(0)}{m!} x^m & \text{if } k \geq j \\ \sum_{m=j-k}^{\infty} \frac{\phi^{(m+k)}(0)}{m!} x^m & \text{otherwise} \end{cases} \quad (27)$$

It follows from the properties of inductive limits (see [12]) that Lemma 1 will be proved if we show that there exists  $p > 1$  such that  $\phi_j \in \tilde{D}_p(\{M_k\})$  for  $j \geq j_0$  and

$$\rho_{I,p}(\phi_j) \rightarrow 0 \quad (28)$$

as  $j \rightarrow \infty$  for every interval  $I$ .

From (13) and (27) we get for  $x \in I$ ,  $k \geq j/2$

$$|\phi_j^{(k)}(x)| \leq \sum_{m=0}^{\infty} \frac{|\phi^{(m+k)}(0)|}{m!} |x|^m \leq A_{I,\phi} h_{\phi}^k \sum_{m=0}^{\infty} \frac{M_{m+k} h_{\phi}^m B_I^m}{m!}.$$

Using 1) and 2), we obtain

$$|\phi_j^{(k)}(x)| \leq A_{I,\phi} h_{\phi}^k \rho(h_{\phi})^{k+1} M_k \sum_{m=0}^{\infty} \tau_m B_I^m \leq \bar{A}_{I,\phi} \bar{h}_{\phi}^k M_k, \quad k \geq j/2, x \in I. \quad (29)$$

In the case  $k < j/2$ ,  $x \in I$  we get

$$|\phi_j^{(k)}(x)| \leq A_{I,\phi} h_{\phi}^k \rho(h_{\phi})^{k+1} M_k \sum_{m:m>j/2} \tau_m B_I^m \leq A'_{I,\phi} (h'_{\phi})^k M_k \sum_{m:m>j/2} \tau_m B_I^m.$$

It follows from the previous inequality and from (29) that there exists a constant  $h_{\phi}$ , depending only on  $\phi$ , and for every interval  $I \subset R^1$  there exists a constant  $A_{I,\phi}$ , depending on  $I$  and  $\phi$ , for which

$$\begin{aligned} \rho_{I,p}(\phi_j) = \sup_{x \in I} \max \{ & \sup_{k:k \geq j/2} [|\phi_j^{(k)}(x)| (p^k M_k)^{-1}], \sup_{k:k \leq j/2} [|\phi_j^{(k)}(x)| (p^k M_k)^{-1}] \leq \\ & A_{I,\phi} \max \{ \sup_{k:k \geq j/2} [h_{\phi}^k p^{-k}], \sup_{k < j/2} [h_{\phi}^k p^{-k} \sum_{m:m>j/2} \tau_m B_I^m] \}. \end{aligned} \quad (30)$$

Using (30), we show that for  $p > h_{\phi}$  condition (28) is satisfied.

This proves Lemma 1.

Let us proceed with the proof of Theorem 4.

Suppose  $\Gamma \in \tilde{D}'(\{M_k\})$ . By Lemma 1, we get for every  $\phi \in \tilde{D}(\{M_k\})$

$$\Gamma(\phi) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} \Gamma(x^k) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(0)}{k!} \mu_k(\Gamma) = \sum_{k=0}^{\infty} (-1)^k \frac{\mu_k(\Gamma)}{k!} \delta_0^{(k)}(\phi).$$

Hence,

$$\Gamma = \sum_{k=0}^{\infty} (-1)^k \frac{\mu_k(\Gamma)}{k!} \delta_0^{(k)} \quad (31)$$

and  $\Gamma$  is a hyperdistribution.

To complete the proof of Theorem 4 we need only to show that

$$\sum_{k=0}^{\infty} \frac{|\mu_k(\Gamma)|}{k!} M_k h^k \leq \infty, \quad h > 0.$$

Fix  $h > 0$  and consider a function  $\phi_h$ , given by the following infinite series:

$$\phi_h(x) = \sum_{j=0}^{\infty} \frac{\text{sign}(\mu_j(\Gamma))}{j!} M_j h^j x^j, \quad x \in R^1.$$

Our goal is to prove that

$$\phi_h \in \tilde{D}(\{M_k\}). \quad (32)$$

If (32) is proved, then

$$\Gamma(\phi_h) = \sum_{j=0}^{\infty} \frac{|\mu_j(\Gamma)| M_j h^j}{j!} < \infty \quad (33)$$

and Theorem 4 will follow from (31) and (33).

We have

$$|\phi_h(x)| \leq \sum_{j=0}^{\infty} \frac{h^j |x|^j M_j}{j!}. \quad (34)$$

From conditions 1)-2) we obtain

$$M_j \leq c(j!) \tau_j M_0, \quad j \geq 0$$

and (34) gives

$$|\phi_h(x)| \leq c M_0 \sum_{j=0}^{\infty} h^j |x|^j \tau_j. \quad (35)$$

Therefore, the series, defining  $\phi_h$ , is uniformly convergent on intervals.

Similarly, the  $k$ -times differentiated series, namely

$$I_k(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \text{sign}(\mu_{j+k}(\Gamma)) M_{j+k} h^{j+k} \quad (36)$$

is uniformly convergent on intervals. This can be shown as follows.

By conditions 1)-2) and by (36), we get

$$|I_k(x)| \leq h^k c^k M_k \sum_{j=0}^{\infty} h^j x^j \tau_j, \quad (37)$$

which shows that the series in (36) converges uniformly.

Hence,

$$\phi_h^{(k)}(x) = I_k(x), \quad x \in R^1. \quad (38)$$

Moreover, from (36), (37), and (38) we get

$$|\phi_h^{(k)}(x)| \leq A_{I,h} \bar{h}^k M_k, \quad k \geq 0, x \in I$$

for every interval  $I$ .

This shows that  $\phi_h \in \tilde{D}(\{M_k\})$ , and hence the proof of Theorem 4 is completed.

The next lemma allows us to construct examples of sequences  $\{M_k\}$ , satisfying conditions 1)-2).

**Lemma 2** *Suppose*

$$M_k = k^{\kappa(k)}, \quad k \geq 0, \quad (39)$$

where  $\kappa(u)$ ,  $u \geq 0$  is a smooth increasing function on  $[0, \infty)$ , for which

$$\kappa'(u) \leq 1 - \varepsilon, \quad u > 0 \quad (40)$$

for some  $\varepsilon$ ,  $0 < \varepsilon < 1$ . Then conditions 1)-2) hold for  $\{M_k\}$ .

**Proof.** It is clear that (40) implies

$$\limsup_{u \rightarrow \infty} \frac{\kappa(u)}{u} \leq 1 - \varepsilon. \quad (41)$$

Denote

$$M(u) = u^{\kappa(u)}, \quad u \geq 0. \quad (42)$$

Then, by (39) and (42), we have  $M_n = M(n)$ ,  $n \geq 0$ .

Now suppose

$$\tau(u) = u^{-\delta u}, \quad u \geq 0,$$

where

$$\delta < \varepsilon/2 \quad (43)$$

is fixed. Then condition 1) with  $\tau_n = \tau(n)$ ,  $n \geq 0$  is satisfied. As for condition 2), it is easy to see that it follows from the inequality

$$M(\mu\lambda)^{1/\lambda} h^\mu \leq \rho(h) \lambda^{(\mu-1)(1-\delta)} (\mu-1)^{(\mu-1)(1-\delta)} M(\lambda)^{1/\lambda}, \quad (44)$$

where  $\lambda \geq 1$ ,  $\mu \geq 1$ ,  $h > h_0$ , and  $M$  is defined in (42). Our goal will be to prove (44).

By (40), we get

$$\frac{\kappa(\lambda\mu)}{\lambda} \leq (1-\varepsilon)(\mu-1) - \frac{\kappa(\lambda)}{\lambda}, \quad (45)$$

and denoting the left-hand side of (45) by  $\Phi_1(\lambda, \mu)$  and the right-hand side by  $\Phi_2(\lambda, \mu)$ , we obtain

$$\lambda^{\Phi_1(\lambda, \mu)} \leq \lambda^{\Phi_2(\lambda, \mu)}. \quad (46)$$

It follows from (41) that

$$\frac{\kappa(u)}{u} \leq 1 - \varepsilon, \quad u \geq u_0. \quad (47)$$

Since

$$\lim_{\mu \rightarrow \infty} \frac{(\mu-1) \ln(\mu-1)}{\mu \ln \mu} = 1,$$

there exists  $\mu_0$ , satisfying

$$\mu_0 > u_0, \quad (48)$$

for which

$$\frac{(\mu-1) \ln(\mu-1)}{\mu \ln \mu} > 1 - \varepsilon, \quad \mu > \mu_0. \quad (49)$$

Here  $v$  is a positive number such that

$$1 - \varepsilon + \delta \leq (1 - \delta)(1 - v). \quad (50)$$

The number  $v$ , satisfying (50), always exists, because, by (43), we have

$$1 - \varepsilon + \delta < 1 - \delta.$$

Suppose

$$\mu^\delta < h.$$

Then

$$h^\mu \mu^{\Phi_1(\mu, \lambda)} \leq h^{\Phi(h, \lambda)}, \quad (51)$$

where

$$\Phi(h, \lambda) = h^{1/\delta} + \frac{1}{\delta\lambda} \kappa(h^{1/\delta} \lambda). \quad (52)$$

For  $h > h_1$ ,  $\lambda \geq 1$  we have

$$\lambda h^{1/\delta} > u_0 \quad (53)$$

Thus, by (47), (52), and (53),

$$\Phi(h, \lambda) \leq h^{1/\delta} + 1/\delta h^{1/\delta}.$$

It follows from (51) that

$$h^\mu \mu^{\Phi_1(\mu, \lambda)} \leq \rho(h), \quad h > h_1, \quad (54)$$

where  $\rho$  is some positive function.

Now suppose

$$h \leq \mu^\delta. \quad (55)$$

Then  $\mu_0^\delta < h$  implies  $\mu_0 < \mu$  and

$$\mu_0 < \lambda\mu. \quad (56)$$

Since

$$(\mu - 1)^{\mu-1} = \mu^{\frac{(\mu-1)\ln(\mu-1)}{\ln \mu}},$$

we have by (49),

$$\mu^{(1-\delta)(1-\nu)\mu} \leq (\mu - 1)^{(1-\delta)(\mu-1)}, \quad \mu > \mu_0. \quad (57)$$

Now it follows from (55), (48), (56), and (47) that

$$h^\mu \mu^{\Phi_1(\mu, \lambda)} \leq \mu^{(\delta+1-\epsilon)\mu}.$$

Moreover, (50) and (57) give

$$h^\mu \mu^{\Phi_1(\mu, \lambda)} \leq (\mu - 1)^{(1-\delta)(\mu-1)}. \quad (58)$$

From (54) and (58) we get

$$h^\mu \mu^{\Phi_1(\mu, \lambda)} \leq \rho(h)(\mu - 1)^{(1-\delta)(\mu-1)}, \quad h > \max(h_1, \mu_0^\delta). \quad (59)$$

Now it is sufficient to multiply the inequalities (46) and (59) and remember the definition of  $\Phi_1$ ,  $\Phi_2$ , and  $M$ . It is easy to see that we get (44) as the result.

Hence, inequality (44) holds and Lemma 2 is proved.

**Corollary 1** *Suppose*

$$M_k = k^{\gamma k}, \quad k \geq 0, \quad 0 < \gamma < 1.$$

*Then Theorem 4 is true for  $\{M_k\}$ .*

Corollary 1 follows easily from Lemma 2 and Theorem 4.



## 5 Classes $D(\{M_k\})$ and the diffusion group

In this section we consider classes  $D(\{M_k\})$ , satisfying the following condition:

$$\lim_{k \rightarrow \infty} \frac{\ln M_k}{k} = \infty. \quad (60)$$

It is easy to see that if condition (60) holds then

$$\lim_{k \rightarrow \infty} \frac{\ln \underline{M}_k}{k} = \infty,$$

where  $\{\underline{M}_k\}$  is the logarithmic regularization of  $\{M_k\}$ , defined in (15).

The following theorem characterizes classes  $D(\{M_k\})$ , which are invariant with respect to the complex-time diffusion group

$$U_z \phi(x) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \phi^{(2k)}(x), \quad x \in R^1, z \in C.$$

**Theorem 5** *Let  $\{M_k\}$  be a sequence, satisfying (60). Then the following assertions are equivalent.*

a)

$$U_z(D(\{M_k\})) \subset D(\{M_k\}), \quad z \in C.$$

b) *There exists a positive function  $\rho(h)$ ,  $h \geq 0$  such that*

$$\underline{M}_{2n+m} h^n \leq \rho(h)^{m+1} n^n \underline{M}_m, \quad m \geq 0, n \geq 0, h > h_0. \quad (61)$$

c) *There exist two positive functions  $\underline{M}$  and  $\rho$  on  $[0, \infty)$  such that*

$$\underline{M}(k) = M_k, \quad k \geq 0$$

*and the inequality*

$$h^\mu \underline{M}(\mu\lambda)^{1/\lambda} \leq \rho(h)(\mu-1)^{\frac{\mu-1}{2}} \lambda^{\frac{\mu-1}{2}} \underline{M}(\lambda)^{1/\lambda} \quad (62)$$

*holds for all  $\lambda \geq 1$ ,  $\mu \geq 1$ , and  $h > h_0$ .*

**Proof.** We begin by showing that if there exists a positive function  $\rho$  such that

$$M_{2n+m}h^n \leq \rho(h)^{m+1}n^n M_m, \quad m \geq 0, n \geq 0, h > h_0, \quad (63)$$

then assertion a) holds.

Suppose (63) is true and let  $\phi \in D(\{M_k\})$ . Then

$$\sum_{k=0}^{\infty} \frac{|z|^k}{k!} |\phi^{(2k+j)}(x)| \leq A_{\phi} h_{\phi}^j \sum_{k=0}^{\infty} \frac{|z|^k}{k!} h_{\phi}^{2k} M_{2k+j}. \quad (64)$$

Applying (63) with  $h = 2h_{\phi}^2|z|$ , we get

$$\frac{M_{2k+j}}{k!} h_{\phi}^{2k} |z|^k \leq 2^{-k} \rho(2h_{\phi}^2|z|)^{j+1} M_j. \quad (65)$$

It follows from (64) and (65) that

$$[U_z \phi(x)]^{(j)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} \phi^{(2k+j)}(x),$$

and

$$|[U_z \phi(x)]^{(j)}| \leq c_{z,\phi} h_{z,\phi}^j M_j, \quad z \in C, j \geq 0,$$

which proves that (63) implies a). Using Cartan-Gorny Theorem (see Theorem 2 in Section 3, we conclude that a) follows from b).

**Remark 3** Analyzing the previous part of the proof of Theorem 3, we see that condition (63) implies not only the validity of inclusion a), but also the continuity of the operators  $U_z$ ,  $z > 0$  on  $D(\{M_k\})$ . Since the topologies of the classes  $D(\{M_k\})$  and  $D(\{\underline{M}_k\})$  coincide (see the proof of Cartan-Gorny Theorem in [8]), the validity of condition b) in Theorem 3 implies the continuity of  $U_z$  on the class  $D(\{M_k\})$ .

The next step consists in proving that a) implies b). Appealing again to Cartan-Gorny Theorem, we see that condition a) is equivalent to the condition

$$U_z(D(\{\underline{M}_k\})) \subset D(\{\underline{M}_k\}), \quad z \in C. \quad (66)$$

Therefore, we should prove that b) follows from (66).

The sequence  $\{\underline{M}_k\}$  is logarithmically convex and hence convex. Without loss of generality we may suppose that  $\underline{M}_k$  increases and that  $\underline{M}_1 = 1$ . It is easy to see that for such  $\{\underline{M}_k\}$  there always exists a smooth logarithmically convex increasing function  $\underline{M}$  on  $[0, \infty)$ , for which

$$\underline{M}(k) = \underline{M}_k, \quad k \geq 0 \quad (67)$$

and

$$\lim_{u \rightarrow \infty} \frac{\ln \underline{M}(u)}{u} = \infty. \quad (68)$$

Now consider the following continuous version of the Ostrovski function (see Section 3)

$$T_1(r) = \sup_u \frac{r^u}{\underline{M}(u)}, \quad r > 0. \quad (69)$$

Denote

$$W_r(u) = \frac{r^u}{\underline{M}(u)}, \quad u > 0, r > 1.$$

Since  $V(0) = 1$  and equality (68) is true, we have

$$W_r(0) = 1, \quad \lim_{u \rightarrow \infty} W_r(u) = 0.$$

Therefore, the continuous function  $W_r$  attains its maximum on  $[0, \infty)$ .

Denote

$$P(u) = \frac{\ln \underline{M}(u)}{u}.$$

Since  $\underline{M}$  is logarithmically convex, the function  $P$  increases.

It is easy to see that the point  $u_0 = u_0(r)$ , at which the maximal value of  $W_r$  is attained, satisfies

$$P(u_0(r)) = \ln r.$$

Thus

$$u_0(r) = P^{-1}(\ln r). \quad (70)$$

It is clear that

$$T_1(r) = \frac{r^{u_0(r)}}{\underline{M}(u_0(r))}, \quad r > 1. \quad (71)$$

After these preliminary considerations we proceed with the proof of the implication  $a) \Rightarrow b)$ .

Suppose (66) holds. Define a function  $\phi$  by

$$\phi(x) = \sum_{k=1}^{\infty} \frac{\cos(\tau(k)x)}{k^2 T_1(\tau(k))}, \quad x \in R^1, \quad (72)$$

where

$$\tau(k) = e^{P(k)}, \quad k \geq 1. \quad (73)$$

Let us show that the series in (72) and all the  $m$ -times termwise differentiated series converge absolutely.

Differentiating formally, we obtain

$$|\phi^{(m)}(x)| \leq \sum_{k=1}^{\infty} \frac{\tau(k)^m}{k^2 T_1(\tau(k))}. \quad (74)$$

From (67) and (69) it is clear that

$$T_1(\tau(k)) \geq \frac{\tau(k)^n}{\underline{M}_n} \quad (75)$$

for every  $k$  and  $n$ . Applying (75) with  $n = m$  to (74), we get

$$|\phi^{(m)}(x)| \leq \underline{M}_m \sum_{k=1}^{\infty} k^{-2}, \quad m \geq 0,$$

which means that  $\phi \in D(\{\underline{M}_k\})$ .

Using (66), we obtain

$$U_{-h}(\phi) \in D(\{\underline{M}_k\}),$$

and recalling the definition of the class  $D(\{\underline{M}_k\})$ , we get

$$|\sum_{k=0}^{\infty} \frac{(-h)^k}{k!} \phi^{(2k+j)}(0)| \leq c_h \bar{\rho}(h)^j \underline{M}_j, \quad j \geq 0 \quad (76)$$

with some positive function  $\bar{\rho}$ .

It is clear that

$$\phi^{(2k+m)}(0) = 0$$

for every odd integer  $m$ . In the case of the even integer  $m = 2j$  we get from (70), (71), and (73) that

$$\phi^{(2k+2j)}(0) = (-1)^{k+j} \sum_{m=1}^{\infty} \frac{\tau(m)^{2k+2j}}{m^2 T_1(\tau(m))} = (-1)^{k+j} \sum_{m=1}^{\infty} \frac{\underline{M}_m \tau(m)^{2k+2j}}{m^2 \tau(m)^m}. \quad (77)$$

It follows from (76) that

$$\sum_{k=0}^{\infty} \frac{h^k}{k!} \sum_{m=1}^{\infty} \frac{\underline{M}_m \tau(m)^{2k+2j}}{m^2 \tau(m)^m} \leq c_h \bar{\rho}(h)^{2j} \underline{M}_{2j}. \quad (78)$$

Taking into account only the term with  $m = 2k + 2j$  in the infinite sum on the left-hand side of (78), we get

$$c_h \bar{\rho}(h)^{2j} \underline{M}_{2j} \geq \sum_{k=0}^{\infty} \frac{h^k}{k!} \frac{\underline{M}_{2k+2j}}{(2k+2j)^2} \geq \sum_{k=0}^{\infty} \frac{h^k}{k!} \frac{\underline{M}_{2k+2j}}{e^{4k+4j}}. \quad (79)$$

It is seen from (79) that assertion b) follows from (66) in the case of an even integer  $m$ . The case of odd  $m$ 's can be treated similarly with only one difference that we take sines instead of cosines in the definition (72) of the function  $\phi$ .

We complete the proof of Theorem 3 by showing that b) is equivalent to c).

Let us first prove that b) is equivalent to the existence of positive functions  $\underline{M}$  and  $\rho$  such that

$$\underline{M}(k) = M_k, \quad k \geq 0 \quad (80)$$

and the inequality

$$\underline{M}(2\xi + \nu)h^\xi \leq \rho(h)^{\nu+1}\xi^\xi \underline{M}(\nu) \quad (81)$$

is true for all  $\xi \geq 0$ ,  $\nu \geq 0$ , and  $h > h_0$ .

It is clear that this assertion implies b). Now suppose b) holds. Consider the function  $\underline{M}$ , defined above in the previous part of the proof. Then inequality (80) holds and, using (80) and (81), we get for  $h > h_0$

$$\underline{M}(2\xi + \nu)h^\xi \leq \underline{M}(2([\xi] + 1) + [\nu] + 1)h^{[\xi]+1} \leq \rho(h)^{[\nu]+2}([\xi] + 1)^{[\xi]+1} \underline{M}_{[\nu]+1}, \quad (82)$$

where  $[a]$  denotes the integer part of  $a$ .

It follows from (82) that

$$\underline{M}(2\xi + \nu)h^\xi \leq \bar{\rho}(h)^{\nu+1}\xi^\xi \underline{M}_{[\nu]+1}. \quad (83)$$

Taking  $n = 1$ ,  $h = 1$  in (61), we obtain

$$\underline{M}_{m+2} \leq c^{m+1} \underline{M}_m, \quad m \geq 0. \quad (84)$$

Now it is clear that (81) follows from (83) and (84). Therefore, (81) is equivalent to b).

Taking  $\xi = \kappa\nu$ ,  $2\kappa + 1 = \mu$ , and  $\nu = \lambda$  in (81) and making simple transformations, we show that (81) is equivalent to assertion c).

Theorem 5 is proved.

**Remark 4** In the first part of the proof of Theorem 5 we showed that condition (63) implies the  $U_z$  - invariance of the class  $D(\{M_k\})$ . Moreover, we did not use condition (60) in this part of the proof.

If we apply Theorem 3, we see that condition (63) for the sequence  $\{M_k\}$ , satisfying (60), implies the similar condition (61) for the sequence  $\{\underline{M}_k\}$ . The inverse assertion does not hold, as it can be easily shown by simple examples. Hence, condition (61) is only sufficient but not necessary for the  $U_z$ -invariance of the class  $D(\{M_k\})$ .

**Corollary 2** Suppose

$$M_k = k^{\kappa(k)}, \quad k \geq 0,$$

where  $\kappa$  is a smooth increasing function on  $(0, \infty)$  such that

1.

$$\limsup_{u \rightarrow \infty} \frac{\kappa(u)}{u} < 1/2.$$

2.

$$\kappa'(u) \leq 1/2, \quad u \neq 0.$$

Then the class  $D(\{M_k\})$  is  $U_z$ -invariant.

**Proof.** It is sufficient to check that inequality (62) with  $M(u) = u^{\kappa(u)}$  instead of  $\underline{M}(u)$  holds. Then (63) will hold and the class  $D(\{M_k\})$  will be  $U_z$ -invariant by Remark 4.

Condition 2) of Corollary 2 gives

$$\kappa(\mu\lambda) \frac{2}{\lambda} \leq \mu - 1 + \kappa(\lambda) \frac{2}{\lambda}.$$

It follows that

$$\lambda^{\kappa(\mu\lambda)2/\lambda} \leq \lambda^{\mu-1+\kappa(\lambda)2/\lambda}. \quad (85)$$

From condition 1) we get

$$\kappa(u) < u/2, \quad u \geq u_0. \quad (86)$$

Thus, there exists  $\tau > 0$  such that

$$\frac{2\kappa(u)}{u} + \tau < 1, \quad u \geq u_0. \quad (87)$$

Fix any  $\xi$  such that  $2\xi \leq \tau$ . Since

$$\lim_{\mu \rightarrow \infty} \frac{(\mu - 1) \ln(\mu - 1)}{\mu \ln \mu} = 1,$$

there exists

$$\mu_0 > u_0, \quad (88)$$

for which

$$\frac{(\mu - 1) \ln(\mu - 1)}{\mu \ln \mu} > 1 - \xi, \quad \mu > \mu_0.$$

Suppose

$$\mu^\xi < h.$$

Then

$$h^\mu \mu^{\kappa(\mu\lambda)2/\lambda} \leq h^{\Phi(h,\lambda)},$$

where

$$\Phi(h, \lambda) = h^{1/\xi} + \frac{2}{\xi\lambda} \kappa(h^{1/\xi}\lambda). \quad (89)$$

For  $h > h_0$ ,  $\lambda \geq 1$  we have

$$\lambda h^{1/\xi} > u_0. \quad (90)$$

Thus, by (86) and (90),

$$2 \sup_{\lambda} \kappa(h^{1/\xi}\lambda) \frac{1}{h^{1/\xi}\lambda} \leq 1. \quad (91)$$



It follows from (89) and (91) that

$$\Phi(h, \lambda) \leq h^{1/\varepsilon} + 1/\xi h^{1/\varepsilon}$$

and

$$h^\mu \mu^{\kappa(\mu\lambda)^{2/\lambda}} \leq \rho(h), \quad h > h_0, \quad (92)$$

where  $\rho$  is some positive function.

Now suppose

$$h \leq \mu^\xi. \quad (93)$$

Then  $h > \mu_0^\xi$  implies  $\mu > \mu_0$  and

$$\lambda\mu > \mu_0. \quad (94)$$

Since

$$(\mu - 1)^{\mu-1} = \mu^{\frac{(\mu-1)\ln(\mu-1)}{\ln \mu}},$$

we have

$$(\mu - 1)^{\mu-1} \geq \mu^{(1-\xi)\mu}, \quad \mu > \mu_0. \quad (95)$$

It follows from (93), (94), (88), and (95) that

$$h^\mu \mu^{\kappa(\mu\lambda)^{2/\lambda}} \leq \mu^{\xi\mu + \kappa(\mu\lambda)^{2/\lambda}} \leq \mu^{\mu(\xi+1-\tau)} \leq \mu^{(1-\xi)\mu} \leq (\mu - 1)^{\mu-1}. \quad (96)$$

From (92) and (96) we get

$$h^\mu \mu^{\kappa(\mu\lambda)^{2/\lambda}} \leq \rho(h)(\mu - 1)^{\mu-1} \quad (97)$$

for  $h > \max(h_0, \mu_0^\xi)$ .

Multiplying inequalities (85) and (97), taking the square roots of the products and recalling the definition of the function  $M$  in Corollary 2, we see that inequality (62) with  $M$

instead of  $\underline{M}$  holds for all  $h > \max(h_0, \mu_0^\xi)$ . As we have already mentioned above, this implies the  $U_z$ -invariance of the class  $D(\{M_k\})$ .

Corollary 2 is established.

The next assertion follows from Corollary 2.

**Corollary 3** Suppose

$$M_k = k^{\frac{k\gamma}{2}}, \quad k \geq 0,$$

where  $0 \leq \gamma < 1$ . Then the class  $D(\{M_k\})$  is  $U_z$ -invariant.

**Corollary 4** If

$$M_k = k^{k/2}, \quad k \geq 0,$$

then the class  $D(\{M_k\})$  is not  $U_z$ -invariant.

Proof. Assume  $D(\{M_k\})$  is  $U_z$ -invariant. Then, by Theorem 5, the inequality (80) should be true. We get in this case

$$\frac{(2n+m)^{n+m/2}}{n!} h^n \leq \rho(h)^{m+1} m^{m/2} \quad (98)$$

for all  $h > h_0$ ,  $n \geq 0$ , and  $m \geq 0$ . Since  $n^n \geq n!$  and  $n < 2n + m$ , we get from (98)

$$(2n+m)^{m/2} h^n \leq \rho(h)^{m+1} m^{m/2}.$$

Fixing  $m$  and allowing  $n$  to tend to infinity, we get a contradiction. Hence, the class  $D(\{M_k\})$  is not  $U_z$ -invariant.

Corollary 4 is proved.

The next theorem is analogous to Theorem 5. It concerns the  $U_z$ -invariance of classes  $D'(\{M_k\})$ . The action of  $\{U_z\}$  on  $D'(\{M_k\})$  is defined by

$$U_z(\Gamma)(\phi) = \Gamma(U_z(\phi)), \quad \Gamma \in D'(\{M_k\}), \phi \in D(\{M_k\}).$$

**Theorem 6** *Let  $\{M_k\}$  be a sequence, satisfying (60). Then the validity of inclusion*

$$U_z(D'(\{M_k\})) \subset D'(\{M_k\})$$

*is equivalent to condition b) or condition c) in Theorem 5.*

Theorem 6 follows easily from Theorem 5 and Remark 3.

## 6 Summary and Conclusions

The purpose of the research carried out under this contract is that of developing the "Chaos Dynamics" approach to the free sheared atmosphere which parallels the successful analysis carried out by Ed Lorenz on the Benard flow (which is a physical model of the troposphere). The basic idea of the Chaos Dynamical analysis is that of (1) expanding the fluid equations in terms of basis functions suited to the geometry and physics of the problem, (2) truncate the expansion to the "lowest" post-linear terms (quadratic in Lorenz' work), (3) deduce an iterative map appropriate to the (strange) attractor given by the truncated post-linear dynamics (the Lorenz "mask" in the case of the Benard problem), and (4) calculate the critical value of the parameter(s) that correspond to both the onset of instability (this critical value can usually be reached by the linear theory) and, most important, to the onset of chaos, which is interpreted as the onset of turbulence. In the case of the free sheared atmosphere the relevant parameter is the Richardson number. Its critical values are at the present not understood. The major portion of our calculations have been carried out for the Taylor- Dyson atmosphere in which both the pressure and the density decrease exponentially with height above the ground and the horizontal shear is given by a Couette flow. This model has been analyzed with fourier analasis which fails to yield unstable

modes even in the most refined forms. Since the standard method fails to give insight into the critical Richardson number, we study an alternative that utilizes modes with finite energy from the start. Our methodology uses hermite polynomials for the infinite interval (tapered by a gaussian) and Laguerre polynomials for the semi- infinite interval (tapered by an exponential). Our expansion method also fails to reveal unstable modes, just like the conventional method. A question of fundamental significance that arises is whether the parameter (scale height) that tapers the polynomials requires such fine adjustment that only a very special choice would correspond to the physical conditions envisioned. We have established that this is not the case and we prove this fact below by showing that taking the scale height to arbitrarily small values does not destroy the convergence of the expansion. In fact the analysis given below establishes with mathematical rigor the validity of our expansion method.

## 7 Acknowledgements

It is a pleasure to acknowledge deep and fruitful discussions with Prof. Robert Hohlfeld, the great help given in many of the calculations by A. Gulisashvili. Special thanks are due to Dr. Robert Beland and to Peter Thomas for introducing us to the problems and for much support.

## 8 References

1. J. A. Dubinskij, "Sobolev Spaces of Infinte Order and Differential Equations," Reidel, Dordrecht et al, 1986.
2. I. M. Gelfand and G. E. Shilov, "Generalized Functions," Academic Press, New York, 1964.
3. I. M. Gelfand and G. E. Shilov, Quelques applications de la théorie des fonctions généralisées, J. Math. Pures et Appl. 35 (1956), 383-413.
4. J. Hadamard, Sur la généralisation de la notion de fonction analytique, Bull. de la Societe Math. Fr. 40 (1912), 28.
5. J. Hadamard, "Lectures on Cauchy's Problem in Linear Partial Differential Equations," Dover, New York, 1952.
6. C. Konstantopoulos, Inverse Problems and Anti-Diffision, Ph.D. Thesis, Boston University, 1992.
7. C. Konstantopoulos, L. Mittag and G. Sandri, Deconvolution of Gaussian filters and antidiffusion, J. Appl. Phys., 68 (1990), 1415-1420.
8. P. Koosis, "The Logarithmic Integral," Cambridge University Press, Cambridge, 1988.
9. M. Lighthill, "Introduction to Fourier Analysis and Generalized Functions," Cambridge University Press, Cambridge, 1970.
10. S. Mandelbrojt, "Séries Adhérentes, Régularization des Suites," Applications, Gauthier-Villars, Paris, 1952.

11. S. Mandelbrojt, "S'eries de Fourier et Classes Quasianalytiques de Fonctions," Gauthier-Villars, Paris, 1935.
12. A. P. Robertson and W. Robertson, "Topological Vector Spaces," Cambridge University Press, Cambridge, 1964.
13. C. Roumieu, Sur quelques extensions de la notion de distribution, Ann. Sci. Ec. Norm. Sup. 77 (1960), 41-121.
14. L. Schwartz, "Theorie des Distributions," Hermann et Cie., Paris, 1951.